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Abstract
We consider the dynamics of the disordered trap model, which is known to
be completely out of equilibrium and to present strong localization effects
in its ageing phase. We are interested in the influence of an external force,
when it is applied from the very beginning at $t = 0$, or only after a waiting
time t_w. We obtain a 'non-linear fluctuation theorem' for the corresponding
one-time and two-time diffusion fronts in any given sample, which implies
the following consequences. (i) For fixed times, there exists a linear response
regime, where the fluctuation–dissipation relation or Einstein relation is valid
even in the ageing time sector, in contrast with other ageing disordered systems.
(ii) For a fixed waiting time and fixed external field, the validity of the linear
response regime is limited in time by a characteristic time depending on the
external force. (iii) In the non-linear response regime, there exists a very
simple relation for the asymmetry in the position for the one-time and two-
times disorder averaged diffusion fronts in the presence of an external force, in
contrast to other models of random walks in random media. The present non-
linear fluctuation theorem is a consequence of a special dynamical symmetry
of the trap model.

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1. Introduction

The field of non-equilibrium statistical mechanics is extremely broad, because the non-
equilibrium character of the dynamics may have various origins. For instance, a system
can be out of equilibrium because it is driven in a non-equilibrium steady state by external
boundary conditions or driving forces [1–3] or because it remains forever non-stationary and
presents ageing, such as in coarsening dynamics [4] or in disordered models [5–7].

Although these various classes of systems usually give rise to different questions and
different approaches, it seems nevertheless that a central goal in all these non-equilibrium

systems has been to study the possible generalizations of the fluctuation–dissipation theorem (FDT), which is an essential result of equilibrium statistical mechanics; it relates the response to an external perturbation with the fluctuations existing in the absence of the perturbation.

On one hand, for non-equilibrium steady states, various generalizations of the FDT have been proposed [8–10], as well as the Gallavoti–Cohen fluctuation theorem [11] for deterministic thermostated dynamics under chaoticity assumptions. This fluctuation theorem, which involves the time-reversed dynamics, has then been extended to various stochastic dynamics, such as Langevin dynamics [12], jump Markov processes and diffusion processes [13, 14]. These various fluctuation theorems provide results in the *non-linear* response regime.

On the other hand, in the field of ageing dynamics of disordered models, the proposed generalization of the FDT concerns the linear response regime and consists of introducing a fluctuation–dissipation ratio to describe the modified relation between the two-time response function and the two-time correlation [5–7]. The introduction of this FDT violation ratio for the linear response regime has also been very useful to characterize the coarsening dynamics in non-disordered systems [15–17]. Very recently, a non-linear fluctuation theorem has also been proposed for the ageing dynamics of disordered spin models [18], where however the time-reversal operation used in non-ageing systems [11–14] has to be completely redefined.

Another important generalization of the usual FDT deals with the non-linear response of systems driven far away from thermal equilibrium, but where the initial state before the application of an external force is the thermal equilibrium [19, 20]. (Note that the initial state in the other approaches described above is usually not arbitrary but the Gibbs–Boltzmann distribution in [12] or the stationary measure in [13].) Remarkably, the Jarzynski non-equilibrium work relation [19] has recently become a very useful tool to reconstruct the free-energy from non-equilibrium single-molecule pulling experiments [21].

This short and very incomplete presentation of recent studies of response and fluctuations in non-equilibrium statistical mechanics has already shown that the subject is very rich and very diverse, and that some unifying framework is still lacking in comparison with equilibrium statistical mechanics.

In this paper, we are interested into the response of trap models on a hypercubic lattice in finite dimension d to an external force field. Trap models provide a simple phenomenological mechanism for ageing [22–24]. The one-dimensional version $d = 1$ is moreover interesting on its own, since it appears in various physical applications concerning transport properties in disordered chains [25, 26] or in the dynamics of denaturation bubbles in random DNA sequences [27]. Its ageing and sub-ageing properties in the absence of a bias have been much studied, either by mathematicians [28–30] and by physicists [31, 32]. Recently, the response of the one-dimensional trap model was studied via scaling arguments and numerical simulations in [33], where various regimes were found to depend on the relative values of the two times considered (t_w, t) and the external applied field f, the main results being that the fluctuation–dissipation relation (or Einstein relation) of the linear response regime is valid even in the ageing sector, whereas the response always become non-linear at long times. The aim of this paper is to show that these response properties are actually two direct consequences of a non-linear fluctuation theorem, that we establish by using a special symmetry of the master equation of the trap model. Another important outcome of the non-linear fluctuation theorem is a prediction for the asymmetry induced by the external force on averaged diffusion fronts. We will always discuss the $d = 1$ case to simplify the presentation, but it is clear that the results can be straightforwardly generalized in arbitrary finite dimension d. However, we will not discuss mean-field trap models, where previous studies on the fluctuation–dissipation relation have shown that the results depend on the observable [34], and on the choice of the functional form of the rates [35, 36]. For the trap models on a lattice in an arbitrary finite

dimension that we consider, there are no such ambiguities in the choice of observables and hopping rates, since the natural observable is the position; we are interested in the response of the position to an external bias, and in the thermal fluctuations of the position.

2. Stochastic dynamics of the trap model

In this section, we present the master equation that describes the stochastic dynamics of the trap model in the absence and in the presence of an external field, and we recall some useful notions and properties.

2.1. Trap model without external field

2.1.1. Master equation in a given sample. The dynamics of the one-dimensional trap model is defined by the following master equation for the probability $P_t(n)$ to be at time t on site n [26]

$$\frac{\mathrm{d}P_t(n)}{\mathrm{d}t} = \frac{P_t(n+1)}{2\tau_{n+1}} + \frac{P_t(n-1)}{2\tau_{n-1}} - \frac{P_t(n)}{\tau_n} \tag{1}$$

supplemented by the simple initial condition at $t = 0$

$$P_{t=0}(n) = \delta_{n,0}. \tag{2}$$

This means that, after arriving on site n, the particle spends on site n a random time t distributed with the exponential distribution of mean τ_n

$$f_{\tau_n}(t) = \frac{1}{\tau_n} \, \mathrm{e}^{-\frac{t}{\tau_n}} \tag{3}$$

and then jumps with equal probability $(1/2)$ to one of its two nearest-neighbour sites $(n-1)$ and $(n+1)$.

The trapping times $\{\tau_n\}$ are quenched random variables in the master equation and represent the disorder; at each visit to the site n, the particle is submitted to the same mean trapping time τ_n. A given sample corresponds to a particular realization of the trapping times on the whole line $\{\ldots, \tau_{-k}, \ldots, \tau_0, \tau_1, \ldots, \tau_k, \ldots\}$.

2.1.2. Law for the disorder. The trapping times are usually taken to be independent random variables distributed with a law presenting the algebraic decay [22, 23]

$$q(\tau) \underset{\tau \to \infty}{\propto} \frac{1}{\tau^{1+\mu}}. \tag{4}$$

This choice comes from the interpretation of the trapping times as Arrhenius times associated with energies E_n

$$\tau_n = \mathrm{e}^{\beta E_n} \tag{5}$$

where the distribution of the energies E_n presents the exponential tail

$$\rho(E) \underset{E \to \infty}{\propto} \mathrm{e}^{-\frac{E}{T_g}} \tag{6}$$

to mimic the lowest energies statistics found in the random energy model [37] and the distribution of free energy of states in the replica theory of spin glasses [38]. This choice is, moreover, reinforced [39] by the exponential tail of the Gumbel distribution, which represents one universality class of extreme-value statistics. In this interpretation, the exponent μ in equation (4) is directly related to the temperature $\mu = T/T_g$ and T_g represents a glass transition.

Indeed, for $\mu < 1$, the mean trapping time $\int d\tau \tau q(\tau)$ is infinite and this directly leads to ageing effects. In particular, the characteristic length scale $\xi(t)$ for the position n at large time t follows the sub-diffusive behaviour obtained by various methods [25, 26, 31, 32, 40]

$$\xi(t) \sim t^{\frac{\mu}{1+\mu}} \tag{7}$$

for $\mu < 1$ and not the usual scaling in $t^{1/2}$ of the pure random walk. For more detailed results concerning the localization properties, the averaged diffusion front as well as ageing and sub-ageing properties, we refer the reader to the recent works [31, 32].

2.2. Trap model in the presence of an external field

In the presence of an external force field f, the master equation (1) becomes [26]

$$\frac{dP_t^{(f)}(n)}{dt} = P_t^{(f)}(n+1)\frac{e^{-\beta\frac{f}{2}}}{2\tau_{n+1}} + P_t^{(f)}(n-1)\frac{e^{+\beta\frac{f}{2}}}{2\tau_{n-1}} - P_t^{(f)}(n)\frac{e^{+\beta\frac{f}{2}} + e^{-\beta\frac{f}{2}}}{2\tau_n} \tag{8}$$

where the hopping rates have been chosen to satisfy the detailed balance condition

$$e^{-\beta U(n)} W_{\{n \to n+1\}}^{(f)} = e^{-\beta U_{n+1}} W_{\{n+1 \to n\}}^{(f)} \tag{9}$$

where the total energy U_n contains both the random energy $(-E_n)$ (5) of the trap n and the potential energy $(-fn)$ linear in the position n induced by the external applied field f

$$U_n = -E_n - fn. \tag{10}$$

The condition (9) guarantees that, in finite sizes, the dynamics will converge towards the steady state given by Boltzmann equilibrium $P_{eq}(n) \sim e^{-\beta U(n)}$.

3. Conservation of the mean position in any given sample

As already emphasized in [33], the trap model has a very special property; when there is no external field $f = 0$ the mean position defined by

$$\langle n \rangle_{f=0}(t) \equiv \sum_{n=-\infty}^{+\infty} n P_t^{(f=0)}(n) \tag{11}$$

is a conserved quantity in any given sample. Indeed, the master equation (1) directly yields

$$\frac{d\langle n \rangle_{f=0}(t)}{dt} = 0 \tag{12}$$

for any realization of the trapping times $\{\tau_n\}$. This comes from the fact that the hopping rates only depend on the initial state, and not on the final state. As a consequence, when a particle gets out of any trap, it always jumps with equal probability $(1/2)$ on the right or on the left. So the trap model possesses a *local* symmetry for the jump probabilities in the two directions. This is in contrast with other models of random walks in random media, such as for instance the Sinai model, where this local symmetry does not hold and where, as a consequence, there is a non-trivial mean position in each sample $\langle n \rangle_{f=0}(t) \neq 0$ [41], and it is only after the average over the samples that the symmetry in the two directions is restored, i.e. $\overline{\langle n \rangle_{f=0}(t)} = 0$.

The property (12) is all the more remarkable because the disorder completely breaks the symmetry $n \to -n$ in any given sample. In particular, the diffusion front $P_t(n)$ in a given sample is, of course, not symmetric in $n \to -n$, and it is only the average over the disordered samples that is symmetric in $n \to -n$, because to each realization $\{\tau_n\}$ we may always associate the reversed realization $\{\tau_{-n}\}$.

4. Non-linear response to a bias applied for $t \geqslant 0$

4.1. A non-linear fluctuation theorem

We now consider a given sample of the trap model and we wish to compare the dynamics in the presence of an external field $(-f)$ with respect to the dynamics in the presence of an external field $(+f)$. It is clear from the two associated master equations (8) corresponding to the bias $(+f)$ and $(-f)$ that we have the remarkable simple property

$$\frac{P_t^{(+f)}(n)}{P_t^{(-f)}(n)} = e^{\beta f n}. \tag{13}$$

We may thus express the differences of the mean position for the two cases as

$$\langle n \rangle_{+f}(t) - \langle n \rangle_{-f}(t) \equiv \sum_{n=-\infty}^{+\infty} n \left[P_t^{(+f)}(n) - P_t^{(-f)}(n) \right] = \sum_{n=-\infty}^{+\infty} n(1 - e^{-\beta f n}) P_t^{(+f)}(n). \tag{14}$$

4.2. Fluctuation–dissipation relation in the linear response regime

A first consequence of (14) is that the susceptibility with respect to the force f reads

$$\left[\frac{\mathrm{d}\langle n \rangle_f(t)}{\mathrm{d}f} \right]_{|_{f \to 0}} = \frac{\beta}{2} \langle n^2 \rangle_{f=0}(t) \tag{15}$$

and thus the fluctuation–dissipation relation is valid for fixed t in the limit $f \to 0$ in the form of the Einstein relation

$$\langle n \rangle_f(t) \underset{f \to 0}{\simeq} \frac{\beta f}{2} \langle n^2 \rangle_{f=0}(t). \tag{16}$$

4.3. Breakdown at long times of the linear response

A second consequence of equation (14) is that, for fixed f, the linear regime (16) of the response is valid only if the linearization of the factor $(1 - e^{-\beta f n})$ is consistent. Thus, it yields the following criterion for the validity for the linear response

$$\beta f \xi_{(f=0)}(t) \ll 1 \tag{17}$$

where $\xi_{(f=0)}(t)$ represents the length scale for the displacement at time t in the unbiased trap model and follows the anomalous scaling (7). The linear response is thus valid only up to a characteristic time presenting the following dependence in the force f

$$t_\mu(f) \equiv \left(\frac{1}{\beta f} \right)^{\frac{1+\mu}{\mu}}. \tag{18}$$

4.4. Consequence for the disorder averaged diffusion front

A third consequence of equation (14) deals with the symmetry breaking between $(n) > 0$ and $(-n) < 0$ for the disorder averaged diffusion front in the presence of the external field $f > 0$. To obtain this property, it is convenient to associate with each realization $\mathcal{D} \equiv \{\tau_n\}$ of the disorder the reversed realization $\mathcal{D}^R \equiv \{\tau_n^R\}$ simply defined by $\tau_n^R = \tau_{-n}$ for any n. It is then clear that the dynamics in the presence of the external force $(+f)$ in the sample $\mathcal{D}$ in the coordinate (n) is completely equivalent to the dynamics in the presence of the external force

$(-f)$ in the sample $\mathcal{D}^R$ in the coordinate $(-n)$, i.e. more explicitly we have for the diffusion fronts

$$P_{\mathcal{D}}^{(f)}(n, t) = P_{\mathcal{D}^R}^{(-f)}(-n, t). \tag{19}$$

So the relation (13) between the diffusion fronts in the same disordered sample, in the presence of two opposite external forces $(+f)$ and $(-f)$, may also be rewritten as a relation between the diffusion fronts in the two reversed disordered samples in the presence of the same external force $(+f)$:

$$\frac{P_{\mathcal{D}}^{(+f)}(n, t)}{P_{\mathcal{D}^R}^{(+f)}(-n, t)} = \mathrm{e}^{\beta f n}. \tag{20}$$

Finally, the average over the samples $\mathcal{D}$, which we denote by an overline, immediately yields the simple property

$$\overline{P_t^{(f)}(-n)} = \mathrm{e}^{-\beta f n}\, \overline{P_t^{(f)}(n)} \tag{21}$$

for the symmetry breaking between $(n) > 0$ and $(-n) < 0$ in the disorder averaged diffusion front in the presence of the external field $f > 0$. As a comparison with other models of random walks in random media, we may cite the Sinai model; the explicit expressions given in [41] for the diffusion front in the presence of a bias show that the property (21) is not satisfied for the Sinai model.

5. Non-linear response to a bias applied for $t \geqslant t_w$

We now consider the following ageing experiment [33], which is very usual in the study of disordered systems [5–7]. The system first evolves with no external force $f = 0$ during the interval $[0, t_w]$, and then evolves in the presence of the external force (f) for $t \geqslant t_w$. We consider the two-time diffusion front $P^{(f;t_w)}(n, t; n_w, t_w|0, 0)$ which represents the joint probability to be at position n_w at time t_w and at position n at time t (where it is assumed that $t > t_w$). Since the dynamics takes place with no bias up to t_w, it is clear that, for any sample, we have from the property (12)

$$\langle n_w \rangle_{f,t_w} \equiv \sum_n \sum_{n_w} n_w P^{(f;t_w)}(n, t; n_w, t_w|0, 0) = 0 \tag{22}$$

so that the mean position at time t also corresponds to the mean displacement between t_w and t

$$\langle n \rangle_{f,t_w}(t) \equiv \sum_n \sum_{n_w} n P^{(f;t_w)}(n, t; n_w, t_w|0, 0) = \langle (n - n_w) \rangle_{f,t_w}(t). \tag{23}$$

Again, we wish to compare the dynamics in a given sample for the two cases $(+f)$ and $(-f)$. The generalization of equation (13) reads

$$\frac{P^{(+f;t_w)}(n, t; n_w, t_w|0, 0)}{P^{(-f;t_w)}(n, t; n_w, t_w|0, 0)} = \mathrm{e}^{\beta f(n - n_w)} \tag{24}$$

which leads to the following property for the relative mean displacement between the two times t_w and t

$$\langle (n - n_w) \rangle_{+f;t_w}(t) - \langle (n - n_w) \rangle_{-f;t_w}(t)$$

$$\equiv \sum_{n=-\infty}^{+\infty} \sum_{n_w=-\infty}^{+\infty} (n - n_w)(1 - \mathrm{e}^{-\beta f(n - n_w)}) P^{(+f;t_w)}(i, t; i_w, t_w|0, 0). \tag{25}$$

5.1. Fluctuation–dissipation relation in the linear response regime

So for fixed times (t, t_w), there is a linear response at small field f given by

$$\langle (n - n_w) \rangle_{f,t_w}(t) \underset{f \to 0}{\simeq} \frac{\beta f}{2} \langle (n - n_w)^2 \rangle_{f=0}(t) \tag{26}$$

which generalizes the one-time Einstein relation (16).

5.2. Breakdown of the linear response regime

However, for fixed waiting time t_w and fixed external field f, the validity of the linear response is limited in time t by the condition of the validity of the linearization of the factor $(1 - e^{-\beta f(n-n_w)})$ in equation (25), which reads

$$\beta f \xi_{f=0}(t, t_w) \ll 1 \tag{27}$$

where $\xi_{f=0}(t, t_w)$ represents the length scale for the relative displacement $(n - n_w)$ between the times t_w and t in the unbiased trap model. In the ageing regime where $t_w \to \infty, t \to \infty$ with the ratio $(t - t_w)/t_w > 1$ fixed, the relative displacement scales as [33, 42]

$$\xi_{f=0}(t, t_w) \sim (t - t_w)^{\frac{\mu}{1+\mu}}. \tag{28}$$

The validity of the linear response regime is thus again limited in the ageing regime by

$$t - t_w \ll t_\mu(f) \tag{29}$$

in terms of the characteristic time (18).

5.3. Consequence for the two-time disorder averaged diffusion front

The generalization of the relation (19) between the diffusion fronts in two reversed disorder samples $\mathcal{D}$ and $\mathcal{D}^R$ to the two-time ageing experiment reads

$$P_{\mathcal{D}}^{(f;t_w)}(n, t; n_w, t_w|0, 0) = P_{\mathcal{D}^R}^{(-f;t_w)}(-n, t; -n_w, t_w|0, 0). \tag{30}$$

So the relation (24) between the diffusion fronts in the same disordered sample, in the presence of opposite external forces $(+f)$ and $(-f)$, may also be rewritten as a relation between the diffusion fronts in the two reversed disordered samples in the presence of the same external forces $(+f)$

$$\frac{P_{\mathcal{D}}^{(f;t_w)}(n, t; n_w, t_w|0, 0)}{P_{\mathcal{D}^R}^{(f;t_w)}(-n, t; -n_w, t_w|0, 0)} = e^{\beta f(n-n_w)}. \tag{31}$$

The average over the samples $\mathcal{D}$ thus yields the following generalization of equation (21) for the asymmetry in n induced by the external force f on the two-time disorder averaged diffusion front

$$\overline{P^{(f)}(-n, t; -n_w, t_w|0, 0)} = e^{-\beta f(n-n_w)} \overline{P^{(f)}(n, t; n_w, t_w|0, 0)}. \tag{32}$$

6. General form of the special symmetry leading to a simple non-linear fluctuation theorem

In this section, we investigate the precise condition under which a general master equation will yield a simple non-linear fluctuation theorem such as that found for the trap model.

6.1. Master equation with detailed balance in the presence of an external field

We consider a general master equation, where the configuration of the system is denoted by $\mathcal{C}$

$$\frac{\mathrm{d}P_t^{(f)}(\mathcal{C})}{\mathrm{d}t} = \sum_{\mathcal{C}'} P_t^{(f)}(\mathcal{C}')W^{(f)}(\mathcal{C}' \to \mathcal{C}) - P_t^{(f)}(\mathcal{C}) \sum_{\mathcal{C}'} W^{(f)}(\mathcal{C} \to \mathcal{C}') \qquad (33)$$

where the hopping rates in the presence of an external field f linearly coupled to the observable b are chosen as

$$W^{(f)}(\mathcal{C} \to \mathcal{C}') = W^0(\mathcal{C} \to \mathcal{C}')\,\mathrm{e}^{\frac{\beta f}{2}[b(\mathcal{C}')-b(\mathcal{C})]} \qquad (34)$$

to satisfy the detailed balance condition

$$\frac{W^{(f)}(\mathcal{C} \to \mathcal{C}')}{W^{(f)}(\mathcal{C}' \to \mathcal{C})} = \frac{W^0(\mathcal{C} \to \mathcal{C}')}{W^0(\mathcal{C}' \to \mathcal{C})}\,\mathrm{e}^{\beta f[b(\mathcal{C}')-b(\mathcal{C})]}. \qquad (35)$$

We consider the simple initial condition

$$P_{t=0}^{(f)}(\mathcal{C}) = \delta_{\mathcal{C},\mathcal{C}_0} \qquad (36)$$

where $(\mathcal{C}_0)$ is some arbitrary given configuration.

6.2. Symmetry condition for the total leaving rates

Let us now define the auxiliary function

$$Q_t(\mathcal{C}) \equiv P_t^{(f)}(\mathcal{C})\,\mathrm{e}^{\beta f[b(\mathcal{C}_0)-b(\mathcal{C})]}. \qquad (37)$$

This satisfies the initial condition

$$Q_{t=0}(\mathcal{C}) \equiv \delta_{\mathcal{C},\mathcal{C}_0} \qquad (38)$$

and evolves with the master equation

$$\frac{\mathrm{d}Q_t(\mathcal{C})}{\mathrm{d}t} = \sum_{\mathcal{C}'} Q_t(\mathcal{C}')W^0(\mathcal{C}' \to \mathcal{C})\,\mathrm{e}^{\frac{\beta f}{2}[b(\mathcal{C}')-b(\mathcal{C})]} - Q_t(\mathcal{C}) \sum_{\mathcal{C}'} W^0(\mathcal{C} \to \mathcal{C}')\,\mathrm{e}^{\frac{\beta f}{2}[b(\mathcal{C}')-b(\mathcal{C})]}. \qquad (39)$$

We wish to compare with the master equation in the presence of the external field $(-f)$, which reads

$$\frac{\mathrm{d}P_t^{(-f)}(\mathcal{C})}{\mathrm{d}t} = \sum_{\mathcal{C}'} P_t^{(-f)}(\mathcal{C}')W^0(\mathcal{C}' \to \mathcal{C})\,\mathrm{e}^{\frac{\beta f}{2}[b(\mathcal{C}')-b(\mathcal{C})]}$$
$$- P_t^{(-f)}(\mathcal{C}) \sum_{\mathcal{C}'} W^{(0)}(\mathcal{C} \to \mathcal{C}')\,\mathrm{e}^{\frac{\beta f}{2}[b(\mathcal{C})-b(\mathcal{C}')]}. \qquad (40)$$

The 'arriving' hopping rates in the configuration $\mathcal{C}$ are thus identical in equations (39) and (40), but the 'leaving' rates out of the configuration $\mathcal{C}$ are obviously different term by term. The important quantity is actually the total leaving rate out of a configuration $\mathcal{C}$ in the presence of the external field $(+f)$, which reads

$$w_{\text{out}}^{(+f)}(\mathcal{C}) \equiv \sum_{\mathcal{C}'} W^0(\mathcal{C} \to \mathcal{C}')\,\mathrm{e}^{\frac{\beta f}{2}[b(\mathcal{C}')-b(\mathcal{C})]} \qquad (41)$$

which is of course *a priori* different from $w_{\text{out}}^{(-f)}(\mathcal{C})$. This means that, generically, the local energy landscape around the configuration $\mathcal{C}$ will be distorted in different ways by the application of $(+f)$ and by the application of $(-f)$.

So the condition to have the simple relation $P_t^{(-f)} = Q_t$, leading to the following non-linear fluctuation theorem

$$\frac{P_t^{(f)}(\mathcal{C})}{P_t^{(-f)}(\mathcal{C})} = e^{\beta f[b(\mathcal{C}) - b(\mathcal{C}_0)]} \tag{42}$$

is the presence of the special symmetry

$$w_{\text{out}}^{(+f)}(\mathcal{C}) = w_{\text{out}}^{(-f)}(\mathcal{C}) \quad \text{for any configuration } \mathcal{C}. \tag{43}$$

This symmetry is satisfied in the trap model for any configuration in spite of the presence of the disorder, because the hopping rates of the unbiased case depend only on the initial configuration and not on the final configuration

$$W^0(\mathcal{C} \rightarrow \mathcal{C}') = W^0(\mathcal{C}). \tag{44}$$

Thus, the disorder contained in the hopping rates disappears from the symmetry condition (43), which is then satisfied because the observable b, which is the position, can increase or decrease by one with equal probabilities.

6.3. Symmetry condition in the linear regime

At linear order in the external field f, the symmetry condition (43) simplifies into the following condition for the unbiased hopping rates

$$\sum_{\mathcal{C}'} [b(\mathcal{C}') - b(\mathcal{C})] W^0(\mathcal{C} \rightarrow \mathcal{C}') = 0 \qquad \text{for any configuration } \mathcal{C}. \tag{45}$$

This exactly means that the mean value $\langle b \rangle$ is a conserved quantity for the unbiased dynamics and for an arbitrary initial configuration

$$\frac{d\langle b \rangle}{dt} \equiv \sum_{\mathcal{C}} b(\mathcal{C}) \frac{dP_t^{(0)}(\mathcal{C})}{dt} = \sum_{\mathcal{C}} P_t^{(0)}(\mathcal{C}) \left[\sum_{\mathcal{C}'} [b(\mathcal{C}') - b(\mathcal{C})] W^{(0)}(\mathcal{C} \rightarrow \mathcal{C}') \right] \tag{46}$$

which vanishes for an arbitrary set of probabilities $P_t^{(0)}(\mathcal{C})$ only if the condition (45) is fulfilled. For the trap model, this condition corresponds to the conservation of the mean position in any given sample (12).

If the condition (45) is satisfied, then the fluctuation–dissipation relation is valid for fixed t in the limit $f \rightarrow 0$ in the form of the Einstein relation

$$\langle (b - b_0) \rangle_f(t) \underset{f \rightarrow 0}{\simeq} \frac{\beta f}{2} \langle (b - b_0)^2 \rangle_{f=0}(t). \tag{47}$$

6.4. Discussion

It would be interesting to find other examples of stochastic dynamics, which satisfy the dynamical symmetry (43), apart from the trap model considered here. A simple way to satisfy the condition (43) would be to be able to associate to each transition $\mathcal{C} \rightarrow \mathcal{C}'$ another transition $\mathcal{C} \rightarrow \mathcal{C}''$, which has the same transition rate in zero field $W^{(0)}(\mathcal{C} \rightarrow \mathcal{C}'') = W^{(0)}(\mathcal{C} \rightarrow \mathcal{C}')$ and such that the variation of the observable b is exactly the opposite $b(\mathcal{C}'') - b(\mathcal{C}) = -(b(\mathcal{C}') - b(\mathcal{C}))$. It is, however, a very strong property, since it should be true for any starting configuration $\mathcal{C}$, and we have not been able to find another example in the most standard dynamics usually considered in statistical physics.

7. Conclusion

In this paper, we have found that the stochastic dynamics of trap models has a very special symmetry that implies a simple non-linear fluctuation theorem for the one-time and two-time diffusion fronts in any given sample. We have obtained the following consequences. (i) In the linear response regime, the fluctuation–dissipation relation or Einstein relation (26) is valid even in the ageing time sector. (ii) For a fixed waiting time t_w and fixed external field f, the validity of the linear response regime is limited in the time $(t - t_w)$ by the characteristic time $t_\mu(f) \sim (\beta f)^{-\frac{\mu}{1+\mu}}$. (iii) In the non-linear regime, we have obtained very simple relations for the asymmetry in the position n induced by an external force on the one-time and the two-time disorder averaged diffusion fronts.

The results (i) and (ii) are in full agreement with the scaling analysis and the numerical simulations presented in [33], and we hope that our approach in some sense 'explains' why ageing trap models, which are known to present strong localization effects and to be completely out of equilibrium (see, for instance, the effective dynamics described in [32]), nevertheless satisfy the fluctuation–dissipation relation in the whole linear response regime, in contrast with many disordered systems that present a non-trivial violation in the linear response regime [5–7].

In a future work, we will consider again the same problem of the effect of an external force on the trap model with a complementary point of view. Here we have used a special symmetry of the master equation to derive general results which are thus valid in the whole ageing phase $0 < \mu < 1$ (4). In the forthcoming paper [42], we will obtain various exact explicit results characterizing response properties in the high disorder limit $\mu \to 0$, by generalizing the previous real-space renormalization approach for the unbiased trap model [32] to include the presence of an external bias. In particular, we will show that the explicit expressions for the averaged diffusion fronts indeed satisfy the constraints (iii) on the asymmetry in n obtained here by a general argument.

Acknowledgments

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